1. **Solution:** X and Y are independent random variables with binomial distributions $\mathbf{Bin}(r,p)$ and $\mathbf{Bin}(s,p)$ respectively. $Z := X+Y$ has $\mathbf{Bin}(r+s,p)$ distribution. The conditional distribution of X given Z is given by

$$
\begin{aligned}
P(X=x|Z=z) &= \frac{P(X=x,Z=z)}{P(Z=z)} \\
&= \frac{P(X=x,Y=z-x)}{P(Z=z)} \\
&= \frac{\binom{r}{x}\binom{s}{z-x}}{\binom{r+s}{z}}
\end{aligned}
$$

where $\min\{0,z-s\} \leq x \leq r$.

Solution (b):

$$
\begin{aligned}
E(X|Z) &= \sum_{x=\min\{0,z-s\}}^{r} x\frac{\binom{r}{x}\binom{s}{z-x}}{\binom{r+s}{z}} \\
&= \sum_{x=\min\{1,z-s\}}^{r} rz\frac{\binom{r-1}{x-1}\binom{s}{z-x}}{(r+s)\binom{r+s-1}{z-1}} \\
&= z\frac{r}{r+s}
\end{aligned}
$$

Solution (c):

$$
\begin{aligned}
E(X(X-1)|Z) &= \sum_{x=\min\{0,z-s\}}^{r} x(x-1)\frac{\binom{r}{x}\binom{s}{z-x}}{\binom{r+s}{z}} \\
&= \frac{rz}{r+s}\sum_{x=\min\{1,z-s\}}^{r} (x-1)\frac{\binom{r-1}{x-1}\binom{s}{z-x}}{\binom{r+s-1}{z-1}} \\
&= \frac{rz}{r+s}\frac{(r-1)(z-1)}{r+s-1}
\end{aligned}
$$

Therefore,

$$
Var(X|Z) = z\frac{r}{r+s}\frac{s}{r+s}\frac{r+s-z}{r+s-1}
$$

$\square$

2. **Solution:** There are r distinguishable balls and n distinguishable boxes $B_1, B_2, \ldots, B_n$. Each ball is put randomly into one of the Boxes.

$$\frac{X_a X_b}{\sum_{1 \leq i < j \leq n} X_i X_j}$$

has the same distribution for any two distinct a and b. Therefore by the linearity of Expectation, we get that the expectation of the above random variable is $\frac{1}{\binom{n}{2}}$. Therefore,

$$E\left(\frac{\sum_{1 \leq a < b \leq m} X_a X_b}{\sum_{1 \leq i < j \leq n} X_i X_j}\right) = \frac{\binom{m}{2}}{\binom{n}{2}}$$

$\square$

3. **Solution:** In top-to-random shuffle, let N_i be the number of shuffles taken for the initial bottommost card to go from $i-$th bottommost position to $(i+1)-$th bottommost position. Observe that $N_{51} = 1$. The number of shuffles needed for the initial bottommost card to come to the top is $N = N_1 + N_2 + \cdots + N_{51}$. Observe that N_i's are independent random variables. $P(N_i = k) = \frac{i}{51}(1 - \frac{i}{51})^{(k-1)}$ for $k \in \mathbb{N}$.

$$E(N) = \sum_{i=1}^{51} E(N_i) = \sum_{i=1}^{51} \frac{51}{i}.$$

Similarly,

$$Var(N) = \sum_{i=1}^{51} Var(N_i) = \sum_{i=1}^{51} \frac{51^2}{i^2}(1 - \frac{i}{51}).$$

$\square$

4. **Solution (a):** Given M, Z has binomial distribution $\mathbf{Bin}(M, 0.2)$. Therefore

$$E(s^Z|M) = (0.2 * s + 0.8)^M.$$

Solution (b): It is given that M has binomial distribution $\mathbf{Bin}(100, 0.5)$. Therefore

$$E(s^Z) = E(E(s^Z|M)) = E(0.2 * s + 0.8)^M = (0.1 * s + 0.9)^{100}.$$

Therefore Z has binomial distribution $\mathbf{Bin}(100, 0.1)$.

$\square$